Gaussian convergence for stochastic acceleration of N particles in the dense spectrum limit

Yves Elskens

GAUSSIAN CONVERGENCE FOR STOCHASTIC ACCELERATION OF $\mathscr{N}$ PARTICLES IN THE DENSE SPECTRUM LIMIT

YVES ELSKENS

ABSTRACT. The velocity of a passive particle in a one-dimensional wave field is shown to converge in law to a Wiener process, in the limit of a dense wave spectrum with independent complex amplitudes, where the random phases distribution is invariant modulo $\pi/2$ and the power spectrum expectation is uniform. The proof provides a full probabilistic foundation to the quasilinear approximation in this limit. The result extends to an arbitrary number of particles, founding the use of the ensemble picture for their behaviour in a single realization of the stochastic wave field.
Keywords : quasilinear diffusion, weak plasma turbulence, propagation of chaos, wave–particle interaction, stochastic acceleration, Fokker–Planck equation, hamiltonian chaos
PACS : 05.45.-a, 52.35.-g, 41.75.-i, 29.27.-a, 84.40.-x
MSC : 34F05, 60H10, 82C05, 82D10, 60J70, 60K40

preprint submitted for publication

Date: July 9, 2012.

1. INTRODUCTION

We first recall the physical setting in sec. 1.1. Indeed the quasilinear approximation is an ubiquitous scheme for deriving irreversible, diffusion-like equations from many-body dynamics, involving a "propagation of chaos" kind of argument in a system with mean-field behaviour. Original derivations of this approximation are sketched in sec. 1.2. A different, more recent analysis in the framework of hamiltonian chaotic dynamics is recalled in sec. 1.3. As these arguments are well detailed in the literature, we do not reproduce the calculations and proofs but merely highlight their key points. Our main result and particulars of the present work are stressed in sec. 1.4.

1.1. **Physical setting.**

The motion of a particle in the field of many waves is a fundamental process in collisionless plasma physics. Even if the particle motion does not feed back on the wave parameters, viz. for a test particle, undergoing passive transport, this problem still presents open issues. Its most elementary instance, in one space dimension, is also a benchmark for approximation techniques.

This one-dimensional problem describes the motion of a particle in a longitudinal, electrostatic, time-dependent potential. Electrostatic modes occur in various contexts [GoRu95, DMA05], including (i) the non-relativistic regime of Coulomb plasmas, where magnetic fields are negligible ; (ii) particle motion parallel to the applied magnetic field in strongly magnetized plasmas ; (iii) particle motion along the axis of a waveguide, such as traveling wave tubes used as amplifiers in telecommunications. The time dependence of the field leads to the propagation of waves, which are longitudinal : Langmuir waves are the simplest collective modes in plasmas. When it applies (in particular for hot plasmas), the neglect of collisions in particle dynamics within plasmas rests on the long-range nature of Coulomb interaction leading to a mean-field picture in both the Vlasov kinetic equation and the Euler fluid models.

In many situations, the wave field evolution involves a response to the particle motion. However, in some instances the particle feedback on the electrostatic field is negligible, and one may take the field as given. The equations of motion for a particle with charge $\mathfrak{e}$ and mass $\mathfrak{m}$ then read

$$\frac{\mathrm{d}X}{\mathrm{d}t} = V(t) \tag{1}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathfrak{e}}{\mathfrak{m}} E(X(t),t) \tag{2}$$

where the electric field E is a prescribed process. It is convenient to represent E as a Fourier series, $E(x,t) = \sum_m E_m \, \mathrm{e}^{\mathrm{i}(k_m x - \omega_m t + \varphi_m)}$, where amplitudes E_m and phases φ_m may be tunable, while (k_m, ω_m) are given by the waves dispersion relation.[1] The key effect of a single wave with phase velocity $v_{\varphi,m} = \omega_m/k_m$ on a particle with velocity v is a tendency [DEM05] to reduce the relative velocity $|v - v_{\varphi,m}|$, and this effect works best when $\mathfrak{m}(v - v_{\varphi,m})^2 \sim |\mathfrak{e}E_m/k_m|$. The competition between two waves m, m' in attempting this synchronization is measured by the wave overlap parameter

$$s_{m,m'} := 2\left|\frac{\mathfrak{e}}{\mathfrak{m}}\right|^{1/2} \frac{|E_m/k_m|^{1/2} + |E_{m'}/k_{m'}|^{1/2}}{|v_{\varphi,m} - v_{\varphi,m'}|}. \tag{3}$$

When this parameter is small, a particle cannot interact strongly with both waves simultaneously, and the dynamics can be analyzed perturbatively. Actually, the dynamics (1)-(2) is well known to be nonintegrable as soon as there is more than a single wave phase velocity ; the two-wave model is a paradigm of hamiltonian chaos with 1.5 degrees of freedom, with a KAM limit as $s \to 0$, and transition to "large scale chaos" as $s \gtrsim 1$. [Es85, ElEs03]

Denoting by Δv_φ the typical relative velocity of a wave with respect to its nearest neighbours, by $k_{\rm typ}$ a typical wavevector, and by $E_{\rm typ}$ a typical amplitude, the regime of interest for this paper is the *dense spectrum limit*, where a particle is typically influenced significantly by many waves ; in this regime the typical resonance overlap parameter $s = 4\sqrt{|(\mathfrak{e}E_{\rm typ})/(\mathfrak{m}k_{\rm typ})|}/\Delta v_\varphi$ is large. It is then tempting to consider the acceleration in the right hand side of (2) as an approximate white noise, and the particle velocity V as a kind of diffusion process : this is the core of the quasilinear approximation [RoFi61, VVS61, VVS62, DrPi62]. The latter is widely used, in diverse physical contexts, as it is easily implemented and relies on simple ideas, which we comment in the following.

1.2. **Original derivations of the quasilinear approximation.** Classical derivations of the quasilinear approximation in plasma physics textbooks, e.g. [Kad65, GoRu95, HaWa04], start from viewing the motion of the test particle as the transport of a measure $\mathrm{d}\mu = f\,\mathrm{d}x\,\mathrm{d}v$ on (x,v) space (with $f(.,.,t)$ possibly a distribution),

$$\partial_t f + v\,\partial_x f = -\frac{\mathfrak{e}}{\mathfrak{m}}\,E(x,t)\,\partial_v f\,, \tag{4}$$

and begin an iterative solution with respect to E,

$$\begin{aligned} f(x,v,t) \;&=\; f(x-vt,v,0) - \frac{\mathfrak{e}}{\mathfrak{m}}\int_0^t E(x-v(t-t_2),t_2)\partial_v f((x-v(t-t_2),v,t_2)\,\mathrm{d}t_2 \\ &\quad + \left(\frac{\mathfrak{e}}{\mathfrak{m}}\right)^2 \int_0^t \int_0^{t_2} E(x-v(t-t_2),t_2)E(x-v(t_2-t_1),t_1) \\ &\qquad\qquad \partial_v f((x-v(t-t_1),v,t_1)\,\mathrm{d}t_1\,\mathrm{d}t_2\,. \end{aligned} \tag{5}$$

[1] Note that this model differs from stochastic acceleration problems in a random potential, for which the field $E(x,t)$ reduces to a static random $E(x)$.

On performing an x average which highlights the correlation function of the electric field E, one then relies on independence of E from the (slaved, passive, tracer) particle distribution $\mathrm{d}\mu$ to eliminate the first order term, and obtains an integro-differential evolution equation for the x-averaged $\bar{f}$. Then, on considering that the velocity process V must be Markov on time scales longer than the correlation time of E, the equation for $\bar{f}(v,t)$ reduces to

$$\partial_t \bar{f} - \partial_v (D(v)\, \partial_v \bar{f}) = 0 \tag{6}$$

where the velocity-dependent diffusion coefficient

$$D(v) = \frac{\mathfrak{e}^2}{\mathfrak{m}^2} \int_0^\infty \langle E(x,t) E(x - v\tau, t - \tau) \rangle \, \mathrm{d}\tau = \frac{\pi \mathfrak{e}^2}{\mathfrak{m}^2} \int \delta(\omega - kv) \langle |E_k|^2 \rangle \, \mathrm{d}k \tag{7}$$

is determined by the wave field lagrangian autocorrelation, with appropriate averaging $\langle \cdot \rangle$ and assuming that phases φ_m are independent and uniformly distributed. The Fourier form in (7), with a Dirac distribution, obtains in the continuous spectrum limit.

The "appropriate averaging" $\langle \cdot \rangle$ may imply (see e.g. sec. 9.4 in [HaWa04]) that one no longer considers the evolution of test particles velocity distribution $\int f(x,v,t)\, \mathrm{d}x$ in a single realization of the field E but rather the *expectation* of this $\int f(x,v,t)\, \mathrm{d}x$ with respect to the ensemble of wave fields. Such a view pertains to the statistics of particle velocities collected from repeated experiments, but it does not apply a priori to the description of transport in a single realization, as stressed in more general terms e.g. p. 45 in [HaWa04].

This derivation may be criticized (within its own viewpoint) on the ground that, however small the coefficient E may be, the differential operator ∂_v is unbounded for many function spaces. Formal, diagrammatic [Bo62a, Bo62b, Bo65, ThBe73, BrFr74] expansions in $E\, \partial_v$ are therefore less straightforward than they may seem.

An alternative derivation, based on particle motion and E-power expansion, also leads to the diffusion equation (6) via its Langevin counterpart, assuming that the particle velocity is a Markov process and computing the first two moments of its increments [St66]. In this context, the "random phase approximation" is actually invoked so that, *to practical ends*, "lagrangian" (as seen by a test particle) phases $k_m(x - v(t - t_j)) + \varphi_m - \omega_m t_j$, can be considered as independent (uniformly distributed modulo 2π) random variables for any relevant sequence of times t_j and wave indices m, viz. not only at a single time (this is imposed by the very distribution of parameters φ_m) but as if their values were "refreshed" repeatedly. The boldness of such an assumption, akin to the propagation of molecular chaos in gas theory [Kac56, Kac59], fueled the debate on the validity of the quasilinear approximation (as a preamble to the further debate focusing on the self-consistent problem, where wave amplitudes and phases evolve under particle feedback) [CEV90, IXW93, LaPe99].

Mathematically, the "repeated random phase approximation" is valid, under a few more technical conditions [PaKo74], in the limit $\varepsilon \to 0$ after a time rescaling, $\tau = \varepsilon^2 t$, when the field E is *mixing*, in the sense that the process $E(.,t)$ is adapted to a family of σ-algebras $\mathscr{F}_s^t$, $0 \leq s \leq t \leq \infty$, with $\mathscr{F}_{s_1}^{t_1} \subseteq \mathscr{F}_{s_2}^{t_2}$ for $0 \leq s_2 \leq s_1 \leq t_1 \leq t_2 \leq \infty$, with a probability measure $\mathbb{P}$ such that the rate function

$$\rho(t) := \sup_{s \geq 0} \sup_{A \in \mathscr{F}_{s+t}^\infty, B \in \mathscr{F}_0^s} |\mathbb{P}(A|B) - \mathbb{P}(A)| \tag{8}$$

satisfies the condition $\int_0^\infty \sqrt{\rho(t)}\, \mathrm{d}t < \infty$. Typical examples of such mixing processes E are ergodic Markov processes on a compact space [PaKo74], but time-periodic fields as discussed e.g. in Refs [CEV90, BeEs98a, ElEs03, ElPa10] fail to meet the mixing condition.

The use of an adjoint formulation instead of trajectories is generally motivated by the traditional viewpoint of kinetic theory, interested in following many particles (in which case, including the self-consistent dynamics where the evolution of E depends on f, measures provide a natural description, see e.g. ch. I.5 in [Sp91]), by the fact that the Vlasov and diffusion equations are linear for f, and by the familiar description of Markov processes in terms of their generator. Yet a single physical realization of the wave field E acts on a particle distribution quite differently from the way an ensemble of independent realizations would act on a single particle [BeEs98b]. The decorrelation assumption is crucial in claiming that the ensemble may describe a single experiment with many particles. Besides, if the Markov assumption fails, the single-time distribution function $f(x, v, t | x_0, v_0, t_0)$ may fail to describe properly the joint n-time distribution $F(x_1, v_1, t_1 \ldots x_n, v_n, t_n)$. Therefore we revisit the derivation of quasilinear equations from a particle viewpoint, and possibly reach a Markov description in an appropriate limit.

1.3. **Hamiltonian dynamics approach.** This dynamics-based program was significantly advanced by Bénisti and Escande [BeEs97, BeEs98a], who proved the validity of the velocity diffusion picture for the dynamics defined by hamiltonian

$$H = \frac{p^2}{2\mathfrak{m}} + \mathscr{A} \sum_{m=-M}^{M} \cos(q - mt + \varphi_m) \tag{9}$$

in the limit $M^{3/2} \gg \mathscr{A}/\mathfrak{m} \to \infty$, when phases φ_m are independent and uniformly distributed in $[0, 2\pi]$. Their derivation relies on the strong chaos (as $s \to \infty$) in particle dynamics associated with the limit, and on the fact that, at a time t, only waves with a phase velocity such that $|v_\varphi - p(t)/\mathfrak{m}| \lesssim \Delta v_\mathrm{b}$ act strongly (nonperturbatively) on the particle. Waves beyond the "resonance box half-width" $\Delta v_{\mathrm{b}} \sim 5(\mathscr{A}/\mathfrak{m})^{2/3}$ can be eliminated from the dynamics (their overall statistical effect is exponentially small in $|v_\varphi - p/\mathfrak{m}|/\Delta v_{\mathrm{b}}$) by a canonical transformation, so that the velocity process is Markov on scales wider than the resonance box. On the other hand, for shorter time scales, the particle velocity needs a time of the order of unity to sample correlations associated with the discreteness of the frequency spectrum, so that it is chaotic and wanders so much that it eventually moves to another resonance box. Moreover, for short time scales, they show how to relax the assumption that all phases are independent to the requirement that any two phases influence negligibly the particle motion [BeEs97, BeEs98a, ElEs03].

This argument was complemented by the observation that the short-time quasilinear approximation holds for times $0 < t \lesssim D^{-1/3} \ln s$, and that the Markov approximation holds for times $t \gtrsim D^{-1/3}$ [EsEl02, ElEs03] ($D^{-1/3}$ is also related to the Lyapunov time scale for the divergence of microscopic trajectories in a typical wave field [ElEs03]), so that the quasilinear approximation holds for all times in the dense spectrum limit $s \to \infty$.

For technical simplicity, the hamiltonian model (9) involves three restrictions with respect to the original dynamics (1)-(2) : all amplitudes are equal, all wavevectors are equal, and all phase velocities are equally spaced. Bénisti and Escande [BeEs98a] sketch how their arguments can be extended to lift these restrictions. The hamiltonian (9) also stresses the spectrum discreteness time scale, as $\Delta v_\varphi = 1$, which can generate correlations over long times [BeEs97, BeEs98b].

1.4. **Position of this work.** In the present work we extend the approach initiated in [El07, ElPa10] and revisit the Bénisti–Escande result with the language of probability theory. We express the wave field as a sum of $N \to \infty$ independent components per unit frequency interval, so that the overlap parameter s diverges in the limit $N \to \infty$. We first prove in Theorem 3.4 that, in the resulting dense wave spectrum limit $s \to \infty$, the wave field acting on a particle for $0 \leq t \leq 2\pi$ converges in law to the field associated with a "white noise". This enables us to derive Proposition 4.2 and Theorem 4.3 on particle motion.

Proposition 4.2 shows that, for $M \to \infty$, for fixed wave power spectral density with $N \to \infty$ so that $s \to \infty$, the velocity of a single particle in the wave field converges in law to a Wiener process over the time interval $[0, 2\pi]$. While Bénisti and Escande emphasize a hamiltonian dynamical system approach, we focus on the velocity and express our limit theorem as a convergence in distribution result, following essentially from central limit averages on the wave field. The convergence in distribution was clearly understood in [BeEs97, BeEs98a], in particular through the statement that the influence of waves outside a resonance box is only perturbative on the statistical properties of the dynamics (p. 914 in [BeEs98a]). The focus on v is also central to the arguments in [ElEs03] which involve the characteristic function $\Phi(\gamma,t) := \mathbb{E}\exp(\mathrm{i}\gamma(v(t)-v(0)))$.

We also pay attention to the behaviour of an arbitrary number $\mathscr{N}$ of particles moving in the same wave field. Their evolutions are not independent processes for finite $\mathscr{A}$, so that the diffusion equation (6) does not describe the evolution of the empirical distribution $\mathscr{N}^{-1}\sum_{\ell=1}^{\mathscr{N}}\delta(v-p_\ell/\mathfrak{m})$; this was stressed in [BeEs98b]. However, our main result, Theorem 4.3, proves that, in the limit $s\to\infty$, particles do diffuse independently, even in the same wave field[2] – in agreement with the view that they generally are in different resonance boxes. Thereby we extend to a broad class of wave fields the conclusion of [ElPa10], which assumed a wave field generated by Wiener processes (viz. the fields obtained in the dense spectrum limit). This result provides some support to the traditional view that a single realization may, in some cases, be approximated by an ensemble.

This paper is organized as follows. We state our assumptions on the wave field in section 2. These enable a fast proof of the convergence of elementary processes associated with the wave field in section 3. Thanks to Ref. [ElPa10] and the continuous mapping theorem [Kal01], the convergence of the $\mathscr{N}$-particle velocity process to the diffusion limit follows immediately in section 4. We stress the interpretation of our techniques and results in section 5. We conclude with a discussion of open issues.

2. Wave field assumptions

A random variable (r.v.) α is symmetric [Kah85, Kal01] if α and $-\alpha$ have identical distributions. Then $\mathbb{E}\,\alpha^k=0$ for odd k if the expectation exists.

ASSUMPTIONS 2.1 (S2, S4). *Given $M\in\mathbb{N}=\{0,1,2,\dots\}$ and $N\in\mathbb{N}_0=\{1,2,\dots\}$, consider $(2M+1)N$ complex random variables $\alpha_{m,n}=A_{m,n}\,\mathrm{e}^{\mathrm{i}\varphi_{m,n}}$. We say that the $\alpha_{m,n}$'s meet assumptions (S2) if*

(1) *they are independent and symmetric,*
(2) $\mathbb{E}A_{m,n}^2=1$,
(3) $\sup_{m,n}\mathbb{E}A_{m,n}^4\le C_4$ *for some* $C_4>1$.

We say that they meet assumptions (S4) if, moreover, the r.v. $\alpha_{m,n}^2$ is also symmetric.

[2]The contrast between this conclusion and Bénisti–Escande's [BeEs98b] might be attributed to the asymptotic nature of our result, as $s\to\infty$. We do not provide estimates for the "convergence rate" of the empirical distribution to its Fokker-Planck limit.

The additional condition for (S4) may be called "*four-symmetry*" for the r.v. α. Examples are (i) the r.v. $\mathrm{e}^{\mathrm{i}(c+K\pi/2)}$, with fixed c and $\mathbb{P}(K=k)=1/4$ for $k\in\{1,2,3,4\}$, (ii) an isotropic complex r.v., viz. $\alpha = A\,\mathrm{e}^{\mathrm{i}\varphi}$ such that φ is uniform on $[0,2\pi]$ (which corresponds to a Steinhaus sequence $\varphi_{m,n}/(2\pi)$ [Kah85]) and independent from A, and in particular (iii) a standard normal complex r.v. (isotropic, with exponentially distributed A^2).

REMARK 2.2. *The third condition in (S2) is unnecessarily stringent (though being met for many physical cases), and could be relaxed to a Lindeberg-type condition.*

Occasionally we identify $\mathbb{R}^2$ with $\mathbb{C}$ to minimize the amount of notations. We denote by B the standard brownian motion in $C(\mathbb{R}^+,\mathbb{R})$ and by W the standard brownian motion in $C(\mathbb{R}^+,\mathbb{C})$, so that B, $\sqrt{2}\,\Re W$ and $\sqrt{2}\,\Im W$ are independent and identically distributed (i.i.d.).

3. CONVERGENCE OF THE CONTROLLING WAVE PROCESSES

Given N real parameters $\sigma_n\in[0,1]$ $(1\le n\le N)$, we first introduce the N complex-valued processes, for $1\le n\le N$,

$$u_n^M(t) = \frac{1}{\sqrt{2\pi}}\int_0^t \sum_{m=-M}^{M} \alpha_{m,n}\,\mathrm{e}^{-\mathrm{i}(m+\sigma_n)s}\,\mathrm{d}s \tag{10}$$

for $t\in\mathbb{R}$. By construction, u_n^M is analytic for any finite n, M, and $\mathrm{e}^{\mathrm{i}\sigma_n t}\,\mathrm{d}u_n^M/\mathrm{d}t$ is a family $(1\le n\le N)$ of independent 2π-periodic complex processes. In the limit $M\to\infty$, the processes u_n^M lose their smoothness (as, typically, their Fourier coefficients decay slowly), but Proposition 3.2 shows that they almost surely (a.s.) admit a Hölder continuous limit u_n.

Specifically, we characterize the smoothness of a function $y\in C(\mathbb{R},\mathbb{C})$ by its modulus of continuity [Kah85, Kal01],

$$\omega_y :]0,\infty[\to[0,\infty] : h\mapsto \omega_y(h) = \sup_{|t-t'|\le h} |y(t')-y(t)|\,. \tag{11}$$

Our first objective is a gaussian convergence theorem, in the limit $N\to\infty$, for the complex-valued process $U_N = N^{-1/2}\sum_n u_n$. Let

$$U_N^M(t) = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} u_n^M(t) \tag{12}$$

for $t\in\mathbb{R}$. Note that if the σ_n's do not vanish and the $\alpha_{m,n}$ are i.i.d., processes u_n are not i.i.d., but they remain independent with closely related moments.

For $g\in C^1([0,2\pi],\mathbb{C})$, let

$$\hat{g}_{m,n} = (2\pi)^{-1/2}\int_0^{2\pi} g(t)\,\mathrm{e}^{-\mathrm{i}(m+\sigma_n)t}\,\mathrm{d}t\,. \tag{13}$$

We also introduce the N complex-valued processes, for $1 \le n \le N$,

$$y_n^M(t) = \frac{1}{\sqrt{2\pi}} \int_0^t \sum_{m=-M}^{M} \alpha_{m,n}\, \mathrm{e}^{-\mathrm{i}ms}\, \mathrm{d}s \tag{14}$$

for $t \in \mathbb{R}$. In case the $\alpha_{m,n}$ are i.i.d., the processes y_n^M are i.i.d. for given M.

PROPOSITION 3.1. *Let $\sigma \in \mathbb{R}$. If $y \in C(\mathbb{R},\mathbb{C})$ has a modulus of continuity ω_y, and $u \in C(\mathbb{R},\mathbb{C})$ is defined by $u(t) = \int_0^t \mathrm{e}^{-\mathrm{i}\sigma s}\, \mathrm{d}y(s)$ for $t \in \mathbb{R}$, then its modulus of continuity satisfies $\omega_u(h) \le (1+|\sigma|h)\omega_y(h)$ for $h \ge 0$.*

Proof First note that, for any $t, t' \in \mathbb{R}$,

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\sigma t'}(u(t')-u(t)) &= \int_t^{t'} \mathrm{e}^{-\mathrm{i}\sigma(s-t')}\, \mathrm{d}y(s) \\
&= y(t')-y(t) + \int_t^{t'} (\mathrm{e}^{-\mathrm{i}\sigma(s-t')}-1)\, \mathrm{d}y(s) \\
&= y(t')-y(t) + \Big[(\mathrm{e}^{-\mathrm{i}\sigma(s-t')}-1)(y(s)-y(t))\Big]_t^{t'} \\
&\quad + \int_t^{t'} (y(s)-y(t))\mathrm{i}\sigma\, \mathrm{e}^{-\mathrm{i}\sigma(s-t')}\, \mathrm{d}s
\end{aligned} \tag{15}$$

by integration by parts. The middle term in the right hand side of (15) vanishes, and we estimate the sum using triangle inequality for $t \le t'$,

$$|u(t')-u(t)| = |\mathrm{e}^{\mathrm{i}\sigma t'}(u(t')-u(t))| \le |y(t')-y(t)| + \int_t^{t'} |y(s)-y(t)||\sigma|\, \mathrm{d}s \tag{16}$$

from which the claim follows by definition of the moduli of continuity. □

For $0 < \beta \le 1$ and $p \in \mathbb{N}$, we denote by $C^{p,\beta}(\mathbb{R},\mathbb{C})$ the class of continuous complex-valued functions of a real variable, with p continuous derivatives, such that their p-th derivative is Hölder continuous with exponent β.

PROPOSITION 3.2. *Let u_n^M and y_n^M be defined by (10) and (14) under assumptions (S2). For any $0 < \beta < 1/2$, and for any n, the sequences y_n^M and u_n^M converge a.s. in $C^{0,\beta}(\mathbb{R},\mathbb{C})$ as $M \to \infty$.*

Proof The y statement results immediately from Theorem 3, Sec. 7.4 in [Kah85], as we compute the sums $s_j^2 = \sum_{m=2^j}^{2^{j+1}-1} m^{-2}\, \mathbb{E}A_{m,n}^2 \approx \left[m^{-1}\right]_{2^j-1/2}^{2^{j+1}-1/2} \approx 2^{-j+1/2}$, using the fact that $\mathbb{E}A_{m,n}^2 = 1$.

The u statement then follows from Proposition 3.1. □

PROPOSITION 3.3. *Let u_n^M and y_n^M be defined by (10) and (14) under assumptions (S2). For any $g \in C^1([0,2\pi],\mathbb{R})$, consider the complex random variables $(g, u_n^M) := \int_0^{2\pi} g(t)\, \mathrm{d}u_n^M(t)$.*

(i) For any M, $\mathbb{E}\,(g,u_n^M)=0$, $\mathbb{E}\,(g,u_n^M)^2=\sum_{m=-M}^{M}\hat{g}_{m,n}^2\,\mathbb{E}\,\alpha_{m,n}^2$ *and* $\mathbb{E}\,|(g,u_n^M)|^2=\sum_{m=-M}^{M}\hat{g}_{m,n}^*\hat{g}_{m,n}$. *Moreover,* $\sup_{n,M}\mathbb{E}\,|(g,u_n^M)|^4\leq(2+C_4)\|g\|_2^4$.
(ii) Assume further that the $\alpha_{m,n}$*'s are four-symmetric. Then as* $M\to\infty$*, the complex r.v.'s* (g,u_n^M) *converge a.s. to a r.v.* (g,u_n) *such that* $\mathbb{E}\,(g,u_n)=0$, $\mathbb{E}\,(g,u_n)^2=0$, $\mathbb{E}\,|(g,u_n)|^2=\int_0^{2\pi}g^2(t)\,\mathrm{d}t$*, and* $\sup_n\mathbb{E}\,|(g,u_n)|^4\leq(2+C_4)\|g\|_2^4$.

Proof Calculations are straightforward as the given test function g is continuous and $[0,2\pi]$ is compact :

$$
\begin{aligned}
(17)\ \mathbb{E}\,(g,u_n^M) &= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}g(t)\sum_{m=-M}^{M}\mathbb{E}\,\alpha_{m,n}\,\mathrm{e}^{-\mathrm{i}(m+\sigma_n)t}\,\mathrm{d}t=0\,,\\
\mathbb{E}\,(g,u_n^M)^2 &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}g(t)g(s)\sum_{m=-M}^{M}\mathbb{E}\,\alpha_{m,n}^2\,\mathrm{e}^{-\mathrm{i}(m+\sigma_n)(t+s)}\,\mathrm{d}t\,\mathrm{d}s\\
(18)\qquad &= \sum_{m=-M}^{M}\mathbb{E}\,\alpha_{m,n}^2\hat{g}_{m,n}^2\,,\\
\mathbb{E}\,|(g,u_n^M)|^2 &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}g(t)g(s)\sum_{m=-M}^{M}\mathbb{E}A_{m,n}^2\,\mathrm{e}^{-\mathrm{i}m(t-s)}\,\mathrm{d}t\,\mathrm{d}s\\
(19)\qquad &= \sum_{m=-M}^{M}\mathbb{E}A_{m,n}^2|\hat{g}_{m,n}|^2\,.
\end{aligned}
$$

Given that $\mathbb{E}A_{m,n}^2=1$, the latter expression yields[3] by Parseval's identity

$$
(20)\qquad \mathbb{E}\,|(g,u_n^M)|^2\leq\sum_{m=-\infty}^{\infty}|\hat{g}_{m,n}|^2=\int_0^{2\pi}g^2(t)\,\mathrm{d}t=\|g\|_2^2
$$

with equality in the limit $M\to\infty$. Finally,

$$
\begin{aligned}
\mathbb{E}\,|(g,u_n^M)|^4 &= \mathbb{E}\sum_{m_1,m_2,m_3,m_4}\alpha_{m_1,n}\alpha_{m_2,n}\alpha_{m_3,n}^*\alpha_{m_4,n}^*\hat{g}_{m_1,n}\hat{g}_{m_2,n}\hat{g}_{m_3,n}^*\hat{g}_{m_4,n}^*\\
&= \sum_m\mathbb{E}A_{m,n}^4|\hat{g}_{m,n}|^4+2\sum_{m_1\neq m_2}\mathbb{E}A_{m_1,n}^2\mathbb{E}A_{m_2,n}^2\,|\hat{g}_{m_1,n}|^2\,|\hat{g}_{m_2,n}|^2\\
(21)\qquad &\leq (C_4+2)\,\|g\|_2^4\,,
\end{aligned}
$$

[3]As pointed out by a referee, this argument reduces to Bessel's inequality, when one views u_n^M as a sum of $2M+1$ basis functions, in the Hilbert space (whose elements are stochastic processes u) with scalar product $(u,v)=\mathbb{E}\int_0^{2\pi}u^*(s)v(s)\,\mathrm{d}s$. Our assumptions on the r.v.'s $\alpha_{m,n}$ ensure orthonormality of our basis.

where the first equality follows from the definition of Fourier coefficients $\hat{g}_{m,n}$, the second equality from the known first two moments of α, and the final inequality from the bound C_4 on $\mathbb{E}A^4$. $\square$

Now we can prove our main claim,

THEOREM 3.4. *Under assumption (S4), the process U_N^M defined by (12) converges in distribution to the brownian motion in $C([0,2\pi],\mathbb{C})$ as $N\to\infty$ and $M\to\infty$.*

Proof First, consider the process $U_N = \lim_{M\to\infty} U_N^M$ in $C^{0,\beta}(\mathbb{R},\mathbb{C})$ for any $0<\beta<1/2$. The convergence is a.s. since U_N^M is a finite linear combination of the processes u_n^M. Given any $g\in C^1([0,2\pi],\mathbb{R})$, we show below that the r.v. $Z_N := (g,U_N)$ converges in distribution to a normal r.v. $Z = X+\mathrm{i}Y$ with $\mathbb{E}Z=0$, $\mathbb{E}X^2=\mathbb{E}Y^2=\frac{1}{2}\|g\|_2^2$, $\mathbb{E}(XY)=0$. As C^1 is dense in L^2, the same holds true for $g\in L^2([0,2\pi],\mathbb{R})$, which will imply that the limit (g,U) is the Wiener integral [Nu06].

The first two moments of Z follow easily from the fact that the r.v.'s $\zeta_n = \xi_n + \mathrm{i}\eta_n := (g,u_n)$ are independent. Proposition 3.3 states that $\mathbb{E}\,\xi_n = \mathbb{E}\,\eta_n = 0$, $\mathbb{E}(2\xi_n\eta_n) = \Im\,\mathbb{E}\,(g,u_n)^2 = 0$ and $\mathbb{E}(\xi_n^2-\eta_n^2) = \Re\,\mathbb{E}\,(g,u_n)^2 = 0$. Besides, $\mathbb{E}(\xi_n^2+\eta_n^2) = \mathbb{E}\,|(g,u_n)|^2 = \|g\|_2^2$.

The fourth moment condition implies the Lindeberg condition on the sequence ζ_n (alternatively, one may adapt the standard proof of the central limit theorem via the characteristic function), so that $N^{-1/2}\sum_{n=1}^N \zeta_n$ converges in distribution to a normal complex random variable, by the gaussian convergence theorem (e.g. Theorem 5.12 in [Kal01]). $\square$

REMARK 3.5. *Our statement holds for arbitrary choice of coefficients σ_n, essentially thanks to the fact that, for any σ, functions $(2\pi)^{-1/2}\,\mathrm{e}^{\mathrm{i}(m+\sigma)t}$ form an orthonormal basis of $L^2([0,2\pi],\mathbb{C})$. In the special case where the $\alpha_{m,n}$ are already normal, each u_n is already a Wiener process.*

REMARK 3.6. *In the case where all $\sigma_n=0$, the processes $u_n(t)-\frac{t}{2\pi}u_n(2\pi)$ define 2π-periodic functions in $C^{0,\beta}(\mathbb{R},\mathbb{C})$; their restrictions to $[0,2\pi]$ converge to the brownian bridge (see [Kal01], ch. 13) and the $N^{-1/2}\sum_n \alpha_{m,n}$ converge to i.i.d. normal r.v.'s.*

Bénisti and Escande [Ben95, BeEs97, BeEs98a] consider the case where $\alpha_{m,n}$ is uniformly distributed on the unit circle, and work with $N=1$. Our statements do not formally apply to such a case. But they let their wave amplitude $\mathscr{A}\to\infty$, so that $s\to\infty$ and $\Delta v_{\mathrm{b}}\to\infty$. To keep finite velocity and amplitude scales, we reformulate their case by relabeling the waves with integer-valued index $m' = mN+n$, letting $\sigma_n = n/N$, and rescaling time as $t' = t/N$, so that $m't' = (m+\sigma_n)t$. To address finite t' scales (of interest to them), we need to extend our previous statements to arbitrarily large time ; the following statement is a first step in this direction.

PROPOSITION 3.7. *Under assumption (S4), assume further that $\sigma_n = n/N$. Then the process U_N^M defined by (12) converges in distribution to the brownian motion in $C(\mathbb{R},\mathbb{C})$ as $N \to \infty$ and $M \to \infty$.*

Proof It suffices to prove convergence over an arbitrarily long time interval $[0,2\pi K]$, with $K \in \mathbb{N}_0$. To extend the previous theorem to $K > 1$, consider first a subsequence $N = N'K$ with $N' \to \infty$. Then let $s = Ks'$ and decompose $n = n' + kN'$ with $1 \leq n' \leq N'$ and $0 \leq k \leq K-1$. Note that

$$
\begin{aligned}
(22)\ U_N^M(t) &= N^{-1/2} \sum_{n=1}^{N'K} \frac{1}{\sqrt{2\pi}} \int_0^{t/K} \sum_m \alpha_{m,n}\, \mathrm{e}^{-\mathrm{i}(mK+k+\frac{n'}{N'})s'} K\, \mathrm{d}s' \\
(23)\qquad\qquad &= \sqrt{\frac{K}{N'}} \sum_{n'=1}^{N'} \frac{1}{\sqrt{2\pi}} \int_0^{t/K} \sum_m \sum_{k=0}^{K-1} \alpha_{m,n}\, \mathrm{e}^{-\mathrm{i}(mK+k+\frac{n'}{N'})s'} K\, \mathrm{d}s'
\end{aligned}
$$

where the latter expression is equivalent to a process $\sqrt{K}U_{N'}^{MK}(t/K)$, up to the $K-1$ terms for which $MK < mK + k \leq MK + K - 1$. These $K-1$ terms do not spoil the limit as their contribution vanishes a.s. for $N' \to \infty$, while by Theorem 3.4 the process $\sqrt{K}U_{N'}^{MK}(t/K)$ converges in distribution to $\sqrt{K}W(t/K)$, which is distributed as $W(t)$.

Now, if $N = N'K + k$ with $1 \leq k < K$, the difference $U_N^M - (1+k/N)^{-1/2}U_{N'K}^M$ converges a.s. to zero as $N' \to \infty$, while $\lim_{N'\to\infty}(1+k/N)^{-1/2} = 1$, so that the sequence U_N^M converges like the subsequence $U_{N'K}^M$. □

REMARK 3.8. *In rough paths terms (see e.g.* [Lej09]*, sec. 8.4, and* [FrVi06] *for definitions and notations), Theorem 3.4 corresponds to the natural extension or lift $\mathbf{U}_N$ (with $\mathbf{U}_N^1 = U_N$) converging in distribution to the geometric enhanced brownian motion in $C^{0,\beta}([0,2\pi],G^2(\mathbb{C}))$ for $1/3 < \beta < 1/2$.*

4. PARTICLE MOTION

We now turn to the solution of differential equations with control U_N^M, viz. to the motion of particles in the wave field associated with the u_n^M's. Since the latter functions are C^1, integration against them must be interpreted so that the limit differentials $\mathrm{d}\Re U$, $\mathrm{d}\Im U$ are the Stratonovich ones [WoZa65, Do77, Su78]. This is satisfactory for the physicists applying e.g. diffusion models, but our result goes further : this formulation opens the way to analysing almost every single realization of the underlying noise, which need not be gaussian.

Specifically, the motion of a particle in the prescribed field of electrostatic waves is described by the system

$$\text{(24)} \quad \mathrm{d}q_N^M = \frac{\mathscr{A}}{\mathfrak{m}} p_N^M \,\mathrm{d}t\,,$$

$$\text{(25)} \quad \mathrm{d}p_N^M = N^{-1/2} \sum_{n=1}^{N} \sum_{m=-M}^{M} A_{m,n} \sin(q_N^M(t) - (m+\sigma_n)t + \varphi_{m,n})\,\mathrm{d}t\,,$$

$$\text{(26)} \quad = \sin(q_N^M(t))\,\mathrm{d}\Re U_N^M(t) + \cos(q_N^M(t))\,\mathrm{d}\Im U_N^M(t)\,,$$

$$\text{(27)} \quad q_N^M(0) = q_0\,, \quad p_N^M(0) = p_0\,,$$

where $\mathscr{A}$ is an overall amplitude scale (incorporating $\mathfrak{e}$) for the waves and $\mathfrak{m}$ is the particle mass. We rescaled the particle momentum p by this overall amplitude to construct an appropriate limit below.

REMARK 4.1. *In the special case where $\sigma_n = 0$ for all n, the N-averaging generates gaussian coefficients for the Fourier wave components for each m. In the case where $\sigma_n = n/N$, the wave field has period $2\pi N$, but its sampling over the shorter interval $[0, 2\pi]$ prevents the observer in the limit $N \to \infty$ from distinguishing it from an actual white noise.*

The previous section implies that, in the limit $M \to \infty, N \to \infty$, the equations of motion may be interpreted as

$$\text{(28)} \quad \mathrm{d}Q = \frac{\mathscr{A}}{\mathfrak{m}} P \,\mathrm{d}t\,,$$

$$\text{(29)} \quad \mathrm{d}P = \sin(Q(t)) \circ \mathrm{d}\Re U(t) + \cos(Q(t)) \circ \mathrm{d}\Im U(t)\,,$$

$$\text{(30)} \quad Q(0) = q_0\,, \quad P(0) = p_0\,,$$

where $\circ\,\mathrm{d}$ denotes the Stratonovich differential [WoZa65].

PROPOSITION 4.2. *In the limit $M \to \infty, N \to \infty$, the process (q_N^M, p_N^M), defined by (24)-(25)-(27) under assumption (S4) with $(q_0, p_0) \in \mathbb{R}^2$, converges in law to (Q, P), where*

$$\text{(31)} \quad Q(t) = q_0 + \frac{\mathscr{A}}{\mathfrak{m}} \big(p_0 t + \int_0^t B(s)\,\mathrm{d}s\big),$$

$$\text{(32)} \quad P(t) = p_0 + B(t)\,,$$

with B the standard one-dimensional Wiener process in $C([0, 2\pi], \mathbb{R})$.

Proof Theorem 3.4 ensures that the limit U is a standard complex Wiener process. Then, for the system (28)-(29)-(30) the mapping $C([0, 2\pi], \mathbb{C}) \to C([0, 2\pi], \mathbb{R}^2) : U \mapsto (P, Q)$ is continuous for the topology of uniform convergence [Do77, Su78], and the continuous mapping theorem [Kal01] transfers the convergence in distribution from the control U to the particle evolution (P, Q).

The proof then follows Ref. [ElPa10]. First note that the Stratonovich solution defined by (28)-(29)-(30) with the Wiener process $(\Re U(t), \Im U(t))$ coincides with the Itô solution because the vector fields $\sin(q)\partial_p$ and $\cos(q)\partial_p$ commute. Finally, since $\cos^2 q + \sin^2 q = 1$, the process defined by $\mathrm{d}P = \sin Q \, \mathrm{d}\Re U(t) + \cos Q \, \mathrm{d}\Im U(t)$ is distributed as the Wiener process in $C([0, 2\pi], \mathbb{R})$. $\square$

Now we turn to the limit $\mathscr{A}/\mathfrak{m} \to \infty$. In this limit, we know that the velocity components of the motions of $\mathscr{N}$ particles also converge jointly in distribution to $\mathscr{N}$ independent Wiener processes. The previous results then imply

THEOREM 4.3. *Given $\mathscr{N}$ initial data (q_0^ℓ, p_0^ℓ) in $\mathbb{R}^2$ ($1 \leq \ell \leq \mathscr{N}$), such that $1 - \cos(q_0^\ell - q_0^{\ell'}) + c|p_0^\ell - p_0^{\ell'}|^2 > 0$ pairwise for some $c > 0$, consider the resulting solutions to (24)-(25)-(27). Then given any $K > 0$, for $\mathscr{A}/\mathfrak{m} \to \infty$, $N \to \infty$, $M \to \infty$, the $\mathscr{N}$-dimensional process p_N^M converges to an $\mathscr{N}$-dimensional Wiener process, and convergence is in law in $C([0, 2\pi K], \mathbb{R}^{\mathscr{N}})$ with the topology of uniform convergence.*

This follows immediately from Theorem 3.1 in [ElPa10], using the brownian limit U and the continuous mapping theorem as in the proof of Proposition 4.2 just given.

5. INTERPRETATION OF THE RESULTS

Our formulation of the limit theorem is rather formal, and our proof strategy differs from the more usual ones in the physics literature.

This paper starts by reducing "noisy wave fields" to "white" ones in the dense spectrum limit, using a *central limit* theorem in the "*wave field* space" of functions U_N^M, as shown in sec. 3. Considering functions u and U is a way to get a handle on the limit process driving the particle motion, while it is harder to define directly the limit in terms of the noise "$\mathrm{d}u/\mathrm{d}t$". The dense spectrum limit is instrumental here to provide the many independent terms in the sum defining the wave field.

The resulting wave field entails the brownian limit for the velocity of a single particle for $0 \leq t \leq 2\pi$ [ElPa10]. This single particle statement makes no reference to any velocity distribution function : we take a "trajectory" viewpoint on stochastic processes, and state a "diffusion process" limit rather than a "Fick equation" limit.

The diffusion picture for $\mathscr{N}$ particles also follows from our previous proof [ElPa10] that, if the wave field is a "periodic white noise", then particles released in the resulting force field are independent in the $\mathscr{A} \to \infty$ limit. This independence between particles results from the fact that particle velocities are a continuous martingale (viz., given the wave field history and their own, their velocity increments have vanishing conditional expectation), from the fact that a martingale is completely characterized by its quadratic variation process (which eliminates the need for considering more than two particles jointly), and from the ergodicity of the random evolution of the relative velocity of any pair of particles. Estimates in [ElPa10] are rather technical, and one may wish to revisit them to provide explicit rates of convergence.

Our order of limits is important : first we take the dense spectrum limit $s \to \infty$, then we let $\mathscr{A} \to \infty$, and finally we consider $\mathscr{N} \geq 1$ and $K \geq 1$, for a single realization of the wave field. Our convergence is in distribution with respect to the wave field random data.

In contrast, usual arguments for the Fokker-Planck limit invoke a loss of memory for the particle motion, directly in terms of particle velocity. The gaussianity of the velocity distribution at a time t (given a Dirac at time 0) is then seen as resulting from a central limit theorem with a sum over (*time*-)successive independent increments. The quantity of interest (see e.g. eq. (9.32) in [HaWa04]) is often the (wave field) ensemble-*averaged* velocity distribution function rather than the empirical distribution driven by a single wavefield.

6. PERSPECTIVES

We proved that the motion of $\mathscr{N}$ particles in the field of random waves approaches a velocity-diffusion process in the dense spectrum limit. Our probabilistic proof highlights a central limit behaviour, while the Bénisti-Escande proof stresses the elimination of correlations by appropriate changes of variables. In comparison with the latter proof, as well as with other derivations of the quasilinear limit, we show that uniformity of phases is unnecessarily strong an assumption : four-symmetry (viz. phase distribution invariant modulo $\pi/2$) is sufficient. We also show that the wave amplitudes need satisfy only rather mild assumptions.

Our proof uses the specific dispersion relation of Bénisti and Escande, $k_m = k$ for all waves, and the regular spacing of phase velocities as $\sigma_n = n/N$. The first assumption enables the decomposition $\sum_m \sin(q - \omega_m t + \varphi_m) = C(t)\sin q + S(t)\cos q$ with coefficients C and S independent of q, and the second one permitted to use the large body of knowledge on random Fourier series. Relaxing these assumptions is physically desirable and will be considered in future work.

In contrast with most earlier works in the plasma physics community, our formulation focuses on full particle trajectories, rather than one-particle distribution functions. In particular, the joint convergence theorem 4.3 supports the familiar picture that the evolution of the empirical distribution

$\mathscr{N}^{-1}\sum_{\ell=1}^{\mathscr{N}}\delta(v-p_\ell/\mathfrak{m})$ a.s. approaches the solution of the diffusion equation $\partial_t f - \partial_v D \partial_v f = 0$ for large $\mathscr{N}$: this *law of large numbers*, and fluctuations around it, require a further limit ($\mathscr{N}\to\infty$) to be discussed in the light of Itô's arguments [Ito83]. In substance, our Theorem 4.3 establishes for velocities what Lebowitz and Spohn [LeSp83] called Assumption C on the motion of particles in position space in order to derive Fick's law for self-diffusion.

Another extension, in the case $\sigma_n = n/N$, would be to allow $K=\kappa N$ with a fixed κ in Theorem 4.3, for it would validate the diffusion picture for times beyond the discretization time $\tau_{\rm disc} := 2\pi/(k\Delta v_\varphi) = 2\pi N$ viz. the time scale after which the wave Fourier spectrum shows its discreteness. While some physical applications of the dense spectrum limit may be viewed as rejecting $\tau_{\rm disc}$ to infinity, the mathematical issue is interesting because of evidence that the diffusion description applies to the single-particle evolution over long times [BeEs97, EsEl03, El10].

ACKNOWLEDGEMENTS

This work benefited from many discussions with D. Escande and members of équipe turbulence plasma, with E. Pardoux, and with participants to the 107th statistical mechanics conference at Rutgers. Stimulating comments by D. Bénisti and D. Escande, and an explanation by A. Lejay are gratefully acknowledged, as are the careful reading and constructive comments by the anonymous referees.

APPENDIX

In Ref. [ElPa10] we introduced the auxiliary process (X_t, Y_t), describing the relative position and velocity of two particles evolving in the same wave field. This process solves

$$\mathrm{d}X_t = Y_t\,\mathrm{d}t \quad , \quad X_0 = x, \tag{33}$$

$$\mathrm{d}Y_t = \sin(X_t)\,\mathrm{d}B_t \quad , \quad Y_0 = y, \tag{34}$$

in the state space $E = \mathbb{T}\times\mathbb{R}\setminus\{(0,0),(\pi,0)\}$, where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and B_t is the standard brownian motion in $C(\mathbb{R}^+,\mathbb{R})$. We proved there in Proposition 5.1 that, for any $(x,y)\in E$, this process a.s. does not reach the points $\{(0,0),(\pi,0)\}$ in finite time. The proof in Ref. [ElPa10] does not identify points modulo 2π for their x component ; one can streamline it as follows.

PROPOSITION .1. *For any* $(x,y)\in E$, $\inf\{t>0 : \sin^2(X_t)+Y_t^2 = 0\} = +\infty$ *a.s., and* $\inf\{\theta>0 : \limsup_{t\to\theta^-}(\sin^2(X_t)+Y_t^2) = +\infty\} = +\infty$ *a.s.*

Proof Let $R_t = \sin^2(X_t)+Y_t^2$ and define $Z_t = \log R_t$. Denote by τ either of these stopping times, corresponding respectively to $Z_t\to-\infty$ and $Z_t\to+\infty$.

Then Itô calculus on $[0,\tau[$ yields

$$\begin{aligned}
(35)\quad \mathrm{d}\sin^2 X_t &= (2\sin X_t\cos X_t)\,Y_t\,\mathrm{d}t\,,\\
(36)\quad \mathrm{d}Y_t^2 &= 2Y_t\sin(X_t)\,\mathrm{d}B_t+\sin^2(X_t)\,\mathrm{d}t\,,\\
\mathrm{d}Z_t &= \frac{2Y_t\sin(X_t)\cos(X_t)+\sin^2(X_t)}{R_t}\,\mathrm{d}t-2\frac{Y_t^2\sin^2(X_t)}{R_t^2}\,\mathrm{d}t\\
(37)\quad & \quad +2\frac{Y_t\sin(X_t)}{R_t}\,\mathrm{d}B_t\,.
\end{aligned}$$

Noting that $2|ab|\le a^2+b^2$ and that $|\cos x|\le 1$ yields the estimates $Y_t^2\sin^2(X_t)\le R_t^2/4$ and $2Y_t\sin(X_t)\cos(X_t)+\sin^2(X_t)\ge -R_t$, so that on the time interval $[0,\tau[$

$$(38)\qquad Z_t\ge Z_0-\frac{3t}{2}+\int_0^t\varphi_s\,\mathrm{d}B_s$$

where $|\varphi_s|\le 1$. This ensures that Z_t is bounded from below on any finite time interval since B_t is bounded. Hence $\inf\{t>0:R_t=0\}=+\infty$ a.s.

Similar upper estimates imply

$$(39)\qquad Z_t\le Z_0+2t+\int_0^t\varphi_s\,\mathrm{d}B_s$$

ensuring that Z_t is bounded from above on any finite time interval. Hence $\inf\{\theta>0:\limsup_{t\to\theta^-}R_t=+\infty\}=+\infty$ a.s. □

The second claim of the present statement does not supersede Lemma 5.4 of Ref. [ElPa10], which proves that Y_t does a.s. not diverge as $t\to\infty$. The present statement only proves that (X_t,Y_t) remains in E for all $t>0$ a.s.

REFERENCES

[Ben95] Bénisti, D., *Validité de l'équation de diffusion en dynamique hamiltonienne*, thèse de doctorat, université de Provence, Marseille, 1995.

[BeEs97] Bénisti, D., and Escande, D.F., *Origin of diffusion in Hamiltonian dynamics*, Phys. Plasmas **4** (1997) 1576–1581.

[BeEs98a] Bénisti, D., and Escande, D.F., *Finite range of large perturbations in Hamiltonian dynamics*, J. Stat. Phys. **92** (1998) 909–972.

[BeEs98b] Bénisti, D., and Escande, D.F., *Nonstandard diffusion properties of the standard map*, Phys. Rev. Lett. **80** (1998) 4871–4874.

[Bo62a] Bourret, R.C., *Propagation of randomly perturbed fields*, Can. J. Phys. **40** (1962) 782–790.

[Bo62b] Bourret, R.C., *Stochastically perturbed fields, with applications to wave propagation in random media*, Nuovo Cimento **26** (1962) 1–31.

[Bo65] Bourret, R.C., *Ficton theory of dynamical systems with noisy parameters*, Can. J. Phys. **43** (1965) 619–639.

[BrFr74] Brissaud, A., and Frisch, U., *Solving linear stochastic differential equations*, J. Math. Phys. **15** (1974) 524–534.

[CEV90] Cary, J.R., Escande, D.F., and Verga, A.D., *Non quasilinear diffusion far from the chaotic threshold*, Phys. Rev. Lett. **65** (1990) 3132–3135.

[Do77] Doss, H., *Liens entre équations différentielles stochastiques et ordinaires*, Ann. Inst. H. Poincaré sect. B **13** (1977) 99–125.

[DMA05] Doveil, F., Macor, A., and Auhmani, Kh., *Wave-particle interaction investigated in a traveling wave tube*, Plasma Phys. Controlled Fus. **47** (2005) A261–A271.

[DEM05] Doveil, F., Escande, D.F., and Macor, A., *Experimental observation of nonlinear synchronization due to a single wave*, Phys. Rev. Lett. **94** (2005) 085003 (4 pp.).

[DrPi62] Drummond, W., and Pines, D., *Nonlinear stability of plasma oscillations*, Nucl. Fusion Suppl. **3** 1049–1057.

[El07] Elskens, Y., *Quasilinear limit for particle motion in a prescribed spectrum of random waves*, Physics AUC **17 (I)** (2007) 109–121.

[El10] Elskens, Y., *Nonquasilinear evolution of particle velocity in incoherent waves with random amplitudes*, Commun. Nonlinear Sci. Numer. Simul. **15** (2010) 10–15.

[ElEs03] Elskens, Y., and Escande, D., **Microscopic dynamics of plasmas and chaos**, IoP Publishing, Bristol, 2003.

[ElPa10] Elskens, Y., and Pardoux, E., *Diffusion limit for many particles in a periodic stochastic acceleration field*, Ann. Appl. Prob. **20** (2010) 2022–2039.

[Es85] Escande, D.F., *Stochasticity in classical hamiltonian systems : universal aspects*, Phys. Rep. **121** (1985) 166–261.

[EsEl02] Escande, D., and Elskens, Y., *Quasilinear diffusion for the chaotic motion of a particle in a set of longitudinal waves*, Acta Phys. Pol. B **33** (2002) 1073–1084.

[EsEl03] Escande, D.F., and Elskens, Y., *Microscopic dynamics of plasmas and chaos : the wave-particle interaction paradigm*, Plasma Phys. Control. Fusion **45** (2003) A115–124.

[FrVi06] Friz, P., and Victoir, N., *A note on the notion of geometric rough paths*, Probab. Theory Relat. Fields **136** (2006) 395–416.

[GoRu95] Goldston, R.J., and Rutherford, P.H., **Introduction to plasma physics**, IoP Publishing, Bristol, 1995.

[HaWa04] Hazeltine R.D., and Waelbroeck F.L., **The framework of plasma physics**, Westview press, Boulder, 2004.

[IXW93] Ishihara, O., Xia, H., and Watanabe, S., *Long-time diffusion in plasma turbulence with broad uniform spectrum*, Phys. Fluids B **5** (1993) 2786–2792.

[Ito83] Itô, K., *Distribution-valued processes arising from independent brownian motions*, Math. Z. **182** (1983) 17–33.

[Kac56] Kac, M., *Foundations of kinetic theory*, pp. 171–197 in **Proc. 3rd Berkely Symp. Math. Stat. Prob.** (ed. Jerzy Neyman) **3**, University of California, 1956.

[Kac59] Kac, M., **Probability and related topics in physical sciences**, American mathematical society, Providence, 1959.

[Kad65] Kadomtsev, B.B., **Plasma turbulence**, transl. L.C. Ronson and M.G. Rusbridge, Academic press, London, 1965.

[Kah85] Kahane, J-P., **Some random series of functions**, 2nd ed., Cambridge university press, Cambridge, 1985.

[Kal01] Kallenberg, O., **Foundations of modern probability**, 2nd ed., Springer, New York, 2001.

[LaPe99] Laval, G., and Pesme, D., *Controversies about quasi-linear theory*, Plasma Phys. Control. Fusion **41** (1999) A239–A246.

[LeSp83] Lebowitz, J.L., and Spohn, H. *Microscopic basis for Fick's law for self-diffusion*, J. Stat. Phys. **28** (1983) 539–556.

[Lej09] Lejay, A., *Yet another introduction to rough paths*, pp. 1–101 in **Séminaire de Proba-bilités XLII** (eds C. Donati-Martin, M. Émery, A. Rouault and C. Stricker), Lect. Notes Math. **1979**, Springer, Berlin, Heidelberg, 2009.

[Nu06] Nualart, D., **The Malliavin calculus and related topics**, 2nd ed., Springer, Berlin, 2006.

[PaKo74] Papanicolaou, G.C., and Kohler, W., *Asymptotic theory of mixing stochastic ordinary differential equations*, Commmu. Pure Appl. Math. **27** (1974) 641–668.

[RoFi61] Romanov, Yu.A., and Filippov, G.F., *The interaction of fast electron beams with longitudinal plasma waves*, J. Eksp. Theoret. Phys. USSR **40** (1961) 123–132, Sov. Phys. JETP **13** (1961) 87–92.

[Sp91] Spohn, H., *Large scale dynamics of interacting particles*, Springer, Berlin, 1991.

[St66] Sturrock, P., *Stochastic acceleration*, Phys. Rev. **141** (1966) 186–191.

[Su78] Sussman, H.J., *On the gap between deterministic and stochastic ordinary differential equations*, Ann. Prob. **6** (1978) 19–41.

[ThBe73] Thomson, J.J., and Benford, G., *Green's function for markovian systems*, J. Math. Phys. **14** (1973) 531–536.

[TsDM91] Tsunoda, S.I., Doveil, F., and Malmberg, J.H., *Experimental test of quasilinear theory*, Phys. Fluids **B3** (1991) 2747–2757.

[VE97] Vanden Eijnden, E., *Some remarks on the quasilinear treatment of the stochastic acceleration problem*, Phys. Plasmas **4** (1997) 1486–1488.

[VVS61] Vedenov, A.A., Velikhov, E.P., and Sagdeev, R.Z., *Nonlinear oscillations of rarified plasma*, Nuclear Fusion **1** (1961) 82–100 ; English abstract, ibid. 145.

[VVS62] Vedenov, A.A., Velikhov, E.P., and Sagdeev, R.Z., *Quasilinear theory of plasma oscillations*, Nuclear Fusion Suppl. **2** (1962) 465–475.

[WoZa65] Wong, E., and Zakai, M., *On the convergence of ordinary integrals to stochastic integrals*, Ann. Math. Statist. **36** (1965) 1560–1564.

EQUIPE TURBULENCE PLASMA, CASE 321, PIIM, UMR 7345 CNRS, AIX-MARSEILLE UNIVERSITÉ, CAMPUS SAINT-JÉRÔME, FR-13397 MARSEILLE CEDEX 13, yves.elskens@univ-amu.fr